$SO(5,5)$ duality in M-theory and generalized geometry

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Abstract

We attempt to reformulate eleven dimensional supergravity in terms of an object that unifies the three-form and the metric and makes the M-theory duality group manifest. This short note deals with the case of where the U-duality group $SO(5,5)$ acts in five spatial dimensions.

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1　Introduction

In [1], a generalized metric was introduced from considering duality from the perspective of the membrane world volume theory. This naturally combined the metric and the three form of eleven dimensional supergravity into one geometric object. The space on which the metric acts will be the usual spacetime extended by n dimensions whose coordinates are p-forms. The details of how many dimensions one needs to add ie. n and which p-forms to use for the coordinates is dependent on several details that we describe in the next section.

This metric is a generalization of the generalized geometry introduced by Hitchin et al [2] and developed in M-theory by Hull [4] and Pacheco and Waldram [5]. A key goal of the work of [1] was to encode the equations of motion of the space on which the duality group acts (when there is no dimensional reduction) in terms of the generalized metric. This then makes the duality group a manifest global symmetry of the action. Note, we are making no assumptions about the space such as the existence of Killing Vectors.

In [1] the duality group was taken to be $SL(5)$ corresponding to the U-duality group of six dimensions. The four dimensional space on which this duality group acts was extended to ten dimensions with the additional six *dual* dimensions having coordinates described by two forms on the four dimensional space.

In this note, we will extend this to five spatial dimensions where the duality group is $SO(5,5)$. In doing so we will have to produce the appropriate generalized metric that combines g_{ij} and C_{ijk} and determine the action that will reduce to the standard action of eleven dimensional supergravity.

The reduction is when we make the *section condition* that the fields are independent of *dual* coordinates. This is a necessary condition for consistency but it is expected that a more general notion of the *section condition* can be constructed and different solutions will correspond to different duality equivalent theories.

Let us examine the analogy with Kaluza Klein theory to describe these ideas. Starting with Einstein Maxwell theory in four dimensions with four dimensional diffeomorphism symmetry and U(1) gauge symmetry one asks whether this can be reproduced by a higher dimensional theory of gravity alone with diffeomorphism symmetry in higher dimensional space. The answer is of course yes. One removes dependence of the metric on the fifth dimension and the theory reduces to Einstein Maxwell with the vector field of electromagnetism being the off diagonal component of the metric and its gauge symmetry a remnant of the five dimensional diffeomorphism. Once this has been constructed it is then tempting to remove the Kaluza Klein reduction and consider the full five dimensional theory.

Now we have in eleven dimensional supergravity with a metric and a 3-form abelian gauge potential, C_3. Can one play the same trick and find an extended space in which a reduction will produce our original theory. No ordinary Kaluza Klein theory can do this. Yet if we construct the extended space to have two form coordinates as one does in generalized geometry then one can write an action on the extended space that will reduce a la Kaluza and Klein to the usual

eleven dimensional supergravity where the C-field is again the off diagonal component of the metric and its gauge transformation a remnant of the diffeomorphisms on the extended space.

Other work along these lines has been done recently by Hohm, Hull and Zwiebach[6] for generalized geometry in string theory and by Hillmann [7] in M-theory for the group E_7 and by West [8] for the relationship with M-theory to the IIA string. The general set up of considering the role of the duality groups in 11-dimensional supergravity has been discussed over a period of years by amongst others [9, 10, 11, 12]. Recent work studying the idea of curvatures in generalized geometry has appeared in [13, 14] and for the Heterotic string in [15].

2   Generalized Metrics and their dynamics

Generalized metrics are metrics on an extended tangent space. In the initial work of Hitchin corresponding to string theory the tangent space was extended as follows:

$$T\Lambda^1(M) \to T\Lambda^1(M) \oplus T\Lambda^{*1}(M) \tag{1}$$

where the addition of the cotangent space $T\Lambda^{*1}(M)$ can be viewed as adding in the space of string windings. In M-theory, the appropriate extension should correspond to windings of membranes and where possible fivebranes.[1] Thus, since we wish to consider the duality group corresponding to five dimensions we must extend the tangent space to include fivebrane modes as follows:

$$T\Lambda^1(M) \to T\Lambda^1(M) \oplus T\Lambda^{*1}(M) \oplus T\Lambda^{*5}(M). \tag{2}$$

The dimension of the extended space is now sixteen dimensional and so we seek a metric on this sixteen dimensional space. We know that the theory possesses $SO(5,5)$ duality symmetry so we are left with the group theoretic problem of finding an object that acts on the **16** dimensional representation of SO(5,5). Note, in string theory the T-duality symmetry is $O(d,d)$ and one might imagine that this is therefore the same; however, in that case the representation is 10 dimensional and the generalized metric acts on the **10** of SO(5,5). Here in M-theory we act on the **16**.

In a future work we will detail the general construction of how one determines this metric for all the relevant groups. For now we simply state the result as coming from a group theoretic construction with the above considerations.

The generalized metric is (upper case Latin indices run from 1 to 16):

$$M_{IJ} = \begin{pmatrix} g_{ab} + \frac{1}{2}C_a{}^{ef}C_{bef} + \frac{1}{16}X_aX_b & \frac{1}{\sqrt{2}}C_a{}^{mn} + \frac{1}{4\sqrt{2}}X_aV^{mn} & \frac{1}{4}g^{-\frac{1}{2}}X_a \\ \frac{1}{\sqrt{2}}C^{kl}{}_b + \frac{1}{4\sqrt{2}}V^{kl}X_b & g^{kl,mn} + \frac{1}{2}V^{kl}V^{mn} & \frac{1}{\sqrt{2}}g^{-\frac{1}{2}}V^{kl} \\ \frac{1}{4}g^{-\frac{1}{2}}X_b & \frac{1}{\sqrt{2}}g^{-\frac{1}{2}}V^{mn} & g^{-1} \end{pmatrix}, \tag{3}$$

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[1]Also for 7 dimensions and above six-brane windings are relevant.

where g_{ab} is the usual metric on the five dimensional space (the lower case latin indices run from 1 to 5); C_{abc} is the three form potential of eleven dimensional supergravity, and $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$ which may be used to raise an antisymmetric pair of indices. We also define:

$$V^{ab} = \frac{1}{6}\epsilon^{abcde}C_{cde} ,\tag{4}$$

with ϵ^{abcde} being the totally antisymmetric tensor and

$$X_a = V^{de}C_{dea} .\tag{5}$$

Note, that the metric (3) is quartic in C_3, and contains terms involving the metric and also terms second order in the inverse metric. It acts on a sixteen dimensional space of which the first 5 dimensions are the usual space dimensions.

We now attempt to reconstruct the dynamical theory out of this generalized metric. Consider the following Lagrangian,

$$L = g^{1/2}\left(\frac{1}{16}M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2}M^{MN}(\partial_N M^{KL})(\partial_L M_{MK}) \right.$$

$$\left. + \frac{3}{128}M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}) - \frac{1}{8}M^{MN}M^{PQ}(M^{RS}\partial_P M_{RS})(\partial_M M_{NQ}) \right)\tag{6}$$

where $\partial_M = \left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y_{ab}}, \frac{\partial}{\partial z} \right)$.

L can then be evaluated using the definition of the generalized metric M_{MN}, described above, in terms of the usual metric and three form.

We have extended our space to allow the encoding of the duality group as a geometric symmetry of the system. Obviously we must restrict the physical dimension of the space back down to five dimensions. Thus we need to supplement the action with a physical *section condition*.

This *section condition* should be a group covariant. One natural choice of condition is simply that all fields are independent of the additional coordinates $\{y_{ab}\}, z$, ie. on all fields

$$\frac{\partial}{\partial y_{ab}} = 0 , \qquad \frac{\partial}{\partial z} = 0 .\tag{7}$$

This is not a covariant condition but it is a sufficient condition. Restricting the fields in this way should then lead to the action reproducing the usual action for the metric and C-field.

A challenge is to determine the $SO(5,5)$ section condition in generality. Note, that for the string this is known; there one simply uses the metric of $SO(5,5)$ to construct a projector that halves the dimension of the space. Here we need to go from sixteen to five dimensions and a simple projection equation of that sort won't suffice.

We then evaluate L in terms of g_{ab} and C_{abc} with the section condition (7). After a long and careful calculation, the result, up to a total derivative, is

$$L = g^{1/2}(R(\gamma) - \frac{1}{48}F^2).$$ (8)

Hence we recover the usual Lagrangian for (the Bosonic sector of) supergravity.

The Lagrangian (6) is a Lagrangian only for the directions in which we allow a duality transformation to act. For the other directions, the kinetic terms maybe written as follows.

$$T = g^{1/2}(-\frac{1}{16}(tr\ \dot{M}^{-1}\dot{M}) - \frac{3}{128}(tr\ M^{-1}\dot{M})^2).$$ (9)

This paper shows that the generalized metrics can be used to formulate the dynamics of the theory. This is an extension of the Kaluza Klein theory where the additional dimensions have p-form coordinates and we have rewritten gravity with a 3-form gauge field in terms of the dynamics of a single generalized metric M_{IJ} acting on the extended space. At this stage, this is simply a rewriting of supergravity. The suggestive thing about Kaluza Klein theory was that once Einstein Maxwell theory was rewritten as a five dimensional theory it inspired the study of higher dimensional theories where the reduction condition was dropped. This is unlikely to make sense for M-theory. In the string theory context the reduction occurs as a particular solution to the section condition which has its origins in the level matching of the closed string theory. What the physical section constraint is in general for M-theory is the most important question for this approach.

3   Acknowledgements

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